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Forcing of convection due to time-dependent heating near threshold

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The forcing of convection due to time-dependent heating is calculated in terms of an inhomogeneous boundary condition on the complex envelope function at the sidewalls of the container. Conditions for eliminating the forcing are derived. Application is made to recent experiments on the timescale for the onset of convection.

1. Introduction

In a geometrically perfect Rayleigh–Bénard cell with no imposed heat currents through the sidewalls there is a sharp transition ('perfect bifurcation') from the conducting state, with no fluid motion, to the convecting state. If the Rayleigh number is held at a constant value above its critical value, the conducting state is unstable. This unstable state will persist indefinitely, however, unless some perturbation forces the convecting state. Three types of perturbations that initiate the convecting state may be identified. Geometric or thermal imperfections are a common source: these will lead to a rounding of the transition with no sharp bifurcation even under static conditions. In the absence of these imperfections the intrinsic thermal noise may initiate the growth of the convecting pattern. The strength of this noise is very weak, but the timescales for the convection to grow to observable strength under such forcing have been estimated to be not excessively long (Ahlers *et al.* 1981). Mechanical vibration of the cell will presumably play a similar role. Finally, the simple operation of changing the Rayleigh number to reach supercritical values will in general induce lateral heat flows in the fluid that in turn drive the convection. This mechanism leaves a perfect bifurcation in static measurements. A crude estimate of this forcing for the case of perfectly conducting sidewalls was given by Ahlers *et al.* (1981). In the present paper we obtain a much better estimate, albeit for stress-free horizontal boundaries, and extend the calculation to the case of general thermal properties of the sidewalls. In addition we show that the forcing may be incorporated into the boundary perturbation scheme of Daniels (1977, 1978). We also demonstrate that the strength of the forcing may be controlled by varying the temperature of the upper and lower plates independently: indeed the forcing vanishes (in the Boussinesq approximation) if the temperature of the lower plate is raised at the same rate as the temperature of the upper plate is decreased.

Our result is in the form of a boundary condition for the amplitude equation (Newell & Whitehead 1969; Segel 1969; Ahlers *et al.* 1981) at the lateral boundary of the Bénard cell. The result applies whenever the amplitude equation is valid, i.e. for slow space and time variation and for Rayleigh numbers close to threshold. Although the effect we consider is somewhat subtle, it controls a number of experimentally important phenomena near threshold, such as the timescale for convective onset. In addition, the forcing we calculate tends to induce particular patterns of convection – namely those with rolls parallel to the sidewalls – which may not be the optimum geometry at the stationary value of the Rayleigh number above threshold. For example in a cylindrical geometry such as that used by Ahlers *et al.* (1981), the forcing initially leads to an axisymmetric pattern of rolls, which ultimately decays to another pattern, of roughly straight rolls.

In §2 the forcing term is obtained from the Boussinesq equations with stress-free horizontal boundaries near threshold. Section 3 contains an application to the onset time measurements of Ahlers *et al.* (1981) and surprisingly good agreement is obtained between experiment and theory. A subsequent paper (Ahlers, Hohenberg & Lücke 1983) will discuss the forcing of externally modulated convection, a situation for which the same ideas apply.

2. Derivation of forcing term

The basic observation that allows a simple treatment of the problem is that the lateral thermal inhomogeneities induced by the time-dependent heating are confined to a healing length of order the plate separation d near each sidewall. On the other hand near the critical Rayleigh number $R_c = \frac{27}{4}\pi^4$ ($\epsilon = (R-R_c)/R_c \ll 1$) the strength of the convection (defined by the conventional envelope function $A(x,t)$; Ahlers *et al.* 1981) varies over the longer lengthscale $\epsilon^{-\frac{1}{2}}d$. For the case of stress-free horizontal boundaries the forcing may therefore be incorporated as an inhomogeneous boundary condition on the envelope function. A similar result was derived by Daniels (1977) for an externally imposed lateral heat flow, and, once the new boundary condition is derived, his analysis of the solutions may be taken over directly to the present problem.

The fluid variables for two-dimensional flow are the horizontal (x) velocity u, the vertical (z) velocity w, the temperature T and the pressure P, and are described by the Oberbeck–Boussinesq equations

$$\dot{u} = \sigma[\partial_x^2 + \partial_z^2]\,u - [w\,\partial_z + u\,\partial_x]\,u - \partial_x P, \tag{1}$$

$$\dot{w} = \sigma T + \sigma[\partial_x^2 + \partial_z^2]\,w - [w\,\partial_z + u\,\partial_x]\,w - \partial_z P, \tag{2}$$

$$\dot{T} = [\partial_x^2 + \partial_z^2]\,T - [w\,\partial_z + u\,\partial_x]\,T, \tag{3}$$

$$0 = \partial_x u + \partial_z w \tag{4}$$

where distance is scaled by the cell depth d, time by d^2/κ_f (with κ_f the thermal diffusivity of the fluid), and temperatures are scaled by $\kappa_f\,\nu/\alpha g d^3$ (with ν the kinematic viscosity, α the expansion coefficient and g the acceleration due to gravity). The Prandtl number is

$$\sigma = \nu/\kappa_f. \tag{5}$$

We impose the boundary conditions at the upper and lower plates: $w = \partial_z u = 0$ at $z = 0, 1$, and $T = T^u(t)$ at $z = 1$, $T = T^{\ell}(t)$ at $z = 0$. The former conditions are the stress-free conditions used for mathematical convenience. The instantaneous Rayleigh number is defined as

$$R(t) = T^{\ell}(t) - T^{u}(t). \tag{6}$$

We allow for a general time dependence for T^u and T^{ℓ}:

$$T^{u,\ell}(t) = \int \frac{d\omega}{2\pi} T^{u,\ell}(\omega)\, e^{-i\omega t}, \tag{7}$$

but will later take the low-frequency limit. We investigate a semi-infinite geometry with liquid confined to the region $x < 0$ by a rigid sidewall of thickness t_w at which $u = w = 0$. The thermal conditions at the sidewall are completely determined by assuming good thermal contact with upper and lower plates, and no heat loss through the outer surface of the sidewall at $x = t_w$, together with the usual assumption of continuity of temperature and heat flux at $x = 0$. These conditions are effectively realized in many experiments and result in a perfect bifurcation in static measurements.

In the time-independent situation and for small ϵ, the fluid variables outside an $O(1)$ healing length near the sidewall are given in terms of a complex envelope function $A(x,t)$ (see Ahlers *et al.* 1981) by

$$w = \tfrac{1}{2}\sqrt{6}\,\pi[A(x,t)\,e^{iq_0 x} + \text{c.c.}]\sin \pi z + O(\epsilon), \tag{8}$$

$$u = \sqrt{3}\,\pi i[A(x,t)\,e^{iq_0 x} - \text{c.c.}]\cos \pi z + O(\epsilon), \tag{9}$$

$$\theta = \tfrac{9}{4}\sqrt{6}\,\pi^3[A(x,t)\,e^{iq_0 x} + \text{c.c.}]\sin \pi z + O(\epsilon), \tag{10}$$

where $q_0 = \pi/\sqrt{2}$ is the critical wavevector and θ is the deviation of the temperature from the conducting profile. The envelope function satisfies the amplitude equation.

$$\tau_0 \dot{A} = \epsilon A + \xi_0^2 A'' - g|A|^2 A, \tag{11}$$

with $\tau_0^{-1} = \frac{3}{2}\pi^2\sigma/(\sigma+1)$, $\xi_0^2 = 8/3\pi^2$ and $g = \frac{1}{2}$ (the dot denotes a time derivative and the prime an x-derivative). The Nusselt number N (dimensionless heat flux) is given by

$$N - 1 = \frac{R_c}{RS}\int |A|^2\, dx\, dy, \tag{12}$$

where S is the area of the container. Note that $|A| \sim \epsilon^{\frac{1}{2}}$, and A varies on the slow lengthscale $\epsilon^{-\frac{1}{2}}\xi_0$ and timescale $\epsilon^{-1}\tau_0$.

The procedure for determining the boundary condition satisfied by the envelope function at the sidewalls was discussed by Brown & Stewartson (1977) and by Daniels (1977). For time-independent heating and the lateral boundary conditions described above, the resulting boundary condition on the envelope function is homogeneous, and at lowest order it is simply

$$A(x=0) = 0. \tag{13}$$

Daniels considered the effect of an imposed lateral heat flow so that

$$u = w = 0, \quad \frac{\partial \theta}{\partial x} = h(z) \quad (x = 0). \tag{14}$$

He found that the boundary condition was determined by the Fourier component h_1 defined by

$$h(z) = \sum_n h_n \sin n\pi z, \tag{15}$$

leading to the inhomogeneous condition

$$A(x=0) = a\, e^{-i\alpha} \tag{16}$$

with
$$a = \frac{2}{9\pi^4\sqrt{3}}h_1, \tag{17}$$

$$\alpha = -\cot^{-1}(2\sqrt{2}). \tag{18}$$

(Equation (17) differs from equation (3.8) of Daniels (1977) owing to the different normalization we have used in (8–11).) The inhomogeneous term serves to force the convection and renders the bifurcation imperfect. We will derive the analogous inhomogeneous term induced by time-dependent temperatures on the upper and lower plates.

As well as leading to forcing terms, a slow time dependence in the heating will slightly change the functional form of the onset solution from (8)–(10). We will neglect these small effects here. In the limit of low frequency and small deviation from threshold $((R(t)-R_c)/R_c \ll 1)$ the envelope function boundary condition is still governed by the $n = 1$ Fourier component of (15) for the temperature distribution at $x = 0$, and we only need the thermal boundary condition replacing (14) for this component.

We define the liquid temperature distribution

$$T = T^{\text{cond}}(z, t) + \theta(x, z, t), \tag{19}$$

with T^{cond} the conducting temperature profile in a laterally infinite liquid, but with the time-dependent upper and lower plate temperatures $T^{\text{u}}(t)$, $T^{\ell}(t)$; the quantity θ is the laterally varying correction. When the temperature from (19) is inserted into (1) and (2), the term T^{cond} is cancelled by the pressure, and only the correction θ drives convection. We expand

$$\theta(x, z, t) = \int \frac{\mathrm{d}\omega}{2\pi} \sum_n \theta_n(x, \omega) \sin n\pi z\, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{20}$$

It is then a simple matter to solve the heat-conduction equation in the wall with $T = T^{\ell}(t)$ *at* $z = 0$ and $T = T^{\text{u}}(t)$ at $z = 1$, and with vanishing heat current at t_{w}. Continuity of the temperature and of the horizontal heat current at $x = 0$ then imposes the condition on the $n = 1$ component at $x = 0$:

$$\theta_1(\omega) + \mu^{-1}(\omega)\, \partial_x \theta_1(\omega) = 2\mathrm{i}\omega\pi(\lambda_1 - 1)\,(\Omega\bar{\Omega})^{-2}[T^{\text{u}}(\omega) + T^{\ell}(\omega)], \tag{21}$$

with
$$\Omega^2 = \pi^2 - \mathrm{i}\omega, \tag{22}$$

$$(\bar{\Omega})^2 = \pi^2 - \lambda_1 \mathrm{i}\omega, \tag{23}$$

$$\mu(\omega) = \bar{\Omega}\, \lambda_2^{-1} \tanh(\bar{\Omega}\, t_{\text{w}}). \tag{24}$$

The thermal parameters appear as $\lambda_1 \equiv \kappa_{\text{f}}/\kappa_{\text{w}}$ and $\lambda_2 \equiv K_{\text{f}}/K_{\text{w}}$, the ratios of the thermal diffusivities and conductivities respectively, of fluid and wall material. Velocities do not enter the condition (21) because of the rigid boundary at $x = 0$.

In the limit of low frequencies $\omega\tau_0 \lesssim \epsilon$ (which is necessary for the validity of the amplitude equation) and for $\lambda_1\omega \ll \pi^2$, (21) may be transformed to a time-local condition:

$$\theta_1(x, t) + \mu^{-1} \partial_x \theta_1(x, t) = \frac{2}{\pi^3}(1 - \lambda_1)\,[\dot{T}^{\text{u}} + \dot{T}^{\ell}], \tag{25}$$

with
$$\mu = \pi\lambda_2^{-1} \tanh(\pi t_{\text{w}}). \tag{26}$$

We now follow the procedure of Daniels (1977) to derive the boundary condition on the envelope function. This is described in detail by Daniels (1977) and by Cross *et al.* (1983), and we merely sketch the argument here for completeness. We assume that the fluid velocities are small near the sidewalls (for $-x \ll \epsilon^{-\frac{1}{2}}$) and solve the coupled linearized equations for θ, u, w. (As stated before we neglect the deviation of T^{cond} from a linear z-dependence in (3), so that $w\,\partial_z T^{\text{cond}}$ is approximated by Rw as in the usual case.) The solutions take the form of a slowly modulated periodic function that will match to the solutions (8)–(10) in the bulk, together with a wall-localized solution decaying over the length $(2\pi)^{-1}$. Imposing the fluid boundary conditions $u = w = 0$, using (25), and eliminating the wall-localized contribution, we find a boundary condition on the envelope function at lowest order of the form $A = a\,\mathrm{e}^{-\mathrm{i}\alpha}$ as in (16) with

$$a(t) = \left(\frac{4}{9\pi^7\sqrt{3}}\right)\frac{\mu}{1+2\mu/\pi}(1-\lambda_1)\,[\dot{T}^{\mathrm{u}}+\dot{T}^{\ell}], \tag{27}$$

$$\alpha = -\cot^{-1}(2\sqrt{2}). \tag{28}$$

Equations (11), (16), (27), (28) are the main results we wish to present. They are exact in the limit of long times and small deviations from threshold (keeping $\epsilon\tau_0/t$ of order unity).† Homogeneous corrections of higher order in ϵ may be calculated as in previous work (Cross *et al.* 1983).

The amplitude equation (11) and inhomogeneous boundary forcing (27), (28) provide a straightforward way of analysing the forcing of convection due to time-dependent heating in various situations. These results may also be used to attempt to eliminate the effects of such forcing so that any intrinsic forcing may be unmasked. We note that if $T^{\mathrm{u}} = -\dot{T}^{\ell}$, i.e the lower plate is warmed at the same rate as the upper plate is cooled (thereby increasing R) the forcing of the onset pattern is eliminated. Also, if $\lambda_1 = 1$, so that the sidewalls and liquid have the same thermal *diffusivities*, the forcing disappears. These last results are not confined to the low-frequency limit, but follow directly from (21).

It is also of interest to consider the growth of the lowest spatial mode for a particular geometry. This mode is driven by the lateral heat flow induced as the Rayleigh number is increased slowly through threshold, by the more common protocol of raising the temperature of the lower plate alone. In this case the partial differential equation (11) becomes an ordinary differential equation for the amplitude of the lowest mode, and the boundary condition (13) is replaced by an additive 'field' driving this amplitude. Consider for example the two-dimensional situation of a fluid confined between infinite lateral sidewalls at $x = \pm L$. The mode that is thus driven has rolls parallel to the sidewalls.‡ The boundary conditions take the form

$$A(\pm L) = a\,\mathrm{e}^{\mp \mathrm{i}\alpha}, \tag{29}$$

with a determined above and the phase α now given by

$$\alpha = \sqrt{\tfrac{1}{2}}\,\pi L - \cot^{-1}(2\sqrt{2}). \tag{30}$$

† The estimate of the forcing presented in equation (C 20) of Ahlers *et al.* (1981) neglected the coupling of the conduction profile to the velocity field. In addition, this expression was only estimated very roughly, so that the result in (C 21) cannot be considered reliable.

‡ The presence of distant transverse sidewalls at $y = \pm M$ would induce rolls parallel to the x-axis, but these would only appear in the bulk of the container in a horizontal diffusion time $\tau_M = M^2$. The estimate given here is thus valid for $t < \tau_M$.

We will consider the growth of the even solution A_e, which in general will become large ($\gg a$) first. It is then convenient to define

$$A_e(x,t) = a(t)\cos\alpha + \sum_{n=1}^{\infty} \cos\frac{(2n-1)\pi x}{2L} A_n(t), \tag{31}$$

satisfying $A_e(\pm L) = a\cos\alpha$ according to (29). The amplitude $A_1 = \bar{A}$ of the first mode then satisfies, in the linear regime, the equation

$$\dot{\bar{A}} = \tau_0^{-1}\bar{\epsilon}(t)\bar{A} + \bar{f}, \tag{32}$$

with $\bar{\epsilon}(t) = \epsilon(t) - \epsilon_c$, the value of ϵ measured from its value $\epsilon_c = \xi_0^2\pi^2/4L^2$ at the onset of the lowest mode in the absence of the boundary forcing. Here

$$\bar{f} = \frac{4}{\pi}\tau_0^{-1}(\bar{\epsilon}+\epsilon_c)\,a\cos\alpha \tag{33}$$

acts as a forcing field. (We remind the reader that $a(t) \propto \dot{\epsilon}$ through (27).) We have neglected terms involving two time derivatives in (32), since these are small for the slow time dependence we are assuming. Note also that only the *linear* part of the amplitude equation need be considered in order to determine the onset time of convection. At later times, the nonlinear terms in (11) are necessary to limit the growth of the initial pattern, but then the forcing term $\bar{f}$ becomes negligible.

To illustrate the resulting behaviour we consider the case of a linear ramp in $\bar{\epsilon}(t)$ investigated by Ahlers *et al.* (1981):

$$\begin{aligned} \bar{\epsilon}(t) &= -\bar{\epsilon}_0 \quad (t < -t_0), \\ \bar{\epsilon}(t) &= \beta t \quad (t > -t_0), \end{aligned} \tag{34}$$

with $t_0 = \bar{\epsilon}_0/\beta$ and $-\bar{\epsilon}_0 < 0$. In this case $a(t)$ is a constant a_0 for $t > -t_0$, and (32), (33) may be integrated directly to give

$$\bar{A} = \frac{4}{\pi}a_0\cos\alpha\left\{-1+\epsilon_c\bar{\beta}^{-\frac{1}{2}}e^{\bar{\beta}t^2/2}\left[\int_{-t_0/\bar{\beta}^{\frac{1}{2}}}^{t/\bar{\beta}^{\frac{1}{2}}} e^{-s^2/2}\,ds\right]\right\}, \tag{35}$$

where $\bar{\beta} = \tau_0^{-1}b$ and we have assumed $A(x) = 0$ for $t < -t_0$. The quantity in square brackets may be replaced by $(2\pi)^{\frac{1}{2}}$ for the times $t, t_0 \gtrsim \bar{\beta}^{\frac{1}{2}}$ of interest. The asymptotic solution (valid for $a_0 \ll \bar{A} \ll (\bar{\epsilon}/g)^{\frac{1}{2}}$) may be seen to be identical with that arising from an effective forcing in (32)

$$\bar{f} = f_{\text{eff}} = \frac{4}{\pi}\tau_0^{-1}\epsilon_c a_0\cos\alpha, \tag{36}$$

and the term in $\bar{\epsilon}$ in (32) does not contribute to the growth. Inserting the expression for a_0 from (27), we find an effective forcing proportional to the ramp rate:

$$f_{\text{eff}} = \bar{f}_1\beta, \tag{37}$$

with

$$\bar{f}_1 = \frac{4}{\pi}\tau_0^{-1}\epsilon_c\left(\frac{4R_c}{9\pi^7\sqrt{3}}\right)\frac{\mu}{1+2\mu/\pi}(1-\lambda_1)\cos\alpha,$$

where μ is given in (26). Inserting values appropriate to the stress-free boundary conditions, we find

$$\bar{f}_1 = \frac{4\sqrt{3}}{\pi^2}\frac{\sigma}{1+\sigma}\frac{1}{L^2}\frac{\mu}{1+2\mu/\pi}(1-\lambda_1)\cos\alpha. \tag{38}$$

3. Application to the experiments of Ahlers *et al.* (1981)

Let us first summarize the experimental results of Ahlers *et al.* (1981) and their interpretation. A ramp in Rayleigh number of the form (34) was applied to a cylindrical cell, and the initial growth of the Nusselt number was quantitatively fit by (32) and (12), with†

$$\bar{f} = \bar{f}^0 + \beta\bar{f}^1, \tag{39}$$

$\bar{f}^1 = 1.4\times10^{-2}$, $\bar{f}^0 = 7\times10^{-4}$. The small extrapolated value $\bar{f}^0$ for $\beta\rightarrow0$ was ascribed to apparatus imperfections, and it is the coefficient $\bar{f}^1$ which we wish to compare with our estimate (38).

In the experiments of Ahlers *et al.* it was found that the mode initially excited was not the one which ultimately stabilized at constant Rayleigh number. The main effect of this complication is in the nonlinear terms in (11) which do not influence the onset time calculation, but some assumption on the form of the initial mode is necessary in order to evaluate the small threshold shift ϵ_c which enters the expression for the parameter $\bar{f}_1$ in our theory. In the interpretation of the experiments, Ahlers *et al.* tentatively assumed that the initial pattern was hexagonal and the final pattern had concentric rolls with zero amplitude at the centre of the cell. A careful analysis of the critical Rayleigh number, and of the static Nusselt number as a function of Rayleigh number, performed subsequently by Behringer & Ahlers (1982), yielded a different interpretation which is more consistent with the forcing we calculate. Their preferred initial state has concentric rolls with finite amplitude at the centre of the cell, and the final state has roughly straight parallel rolls which break the cylindrical symmetry (this interpretation is confirmed by the experiments of Kirchartz *et al.* 1981). As mentioned above, the only effect of this uncertainty on our (linear) calculation is in the value of the threshold shift ϵ_c which enters (36). We use the axisymmetric envelope function discussed by Ahlers *et al.* (1981), but with an amplitude that is finite at the centre of the cell, so the threshold shift is $\epsilon_c = \pi^2\xi_0^2/4L^2$, where L is the radius of the cell. The analysis leading to (38) may be repeated for the cylindrical pattern and the same result is found, once again repeated for stress-free horizontal boundaries.

In attempting to apply our theory to real experiments it is necessary to consider the effect of rigid horizontal boundary conditions. As is well known, the linear solutions near threshold are more complicated than (8)–(10), but an amplitude equation of the form (11) may still be derived (Wesfreid *et al.* 1978; Cross 1980) for a laterally infinite system. Unfortunately, the effect of sidewalls is not simple to analyse in this case even for static situations, so there is no rigorous result analogous to the boundary condition (16), even arbitrarily close to threshold. In the absence of such a derivation, it appears to us reasonable to assume a boundary condition of the form of (16), (27) and (28), and a forcing field analogous to (36) and (39), with unknown $O(1)$ changes in the numerical factors.

The theoretical result in (38) may be evaluated for the experimental conditions of Ahlers *et al.* (1981), where $L = 4.72$, $\sigma = 0.78$, $\lambda_1 \approx 0$, $\mu = \pi^2K_w t_w/K_f = 3.26$, with the result

$$\bar{f}_1 = 0.015\,|\cos\alpha|. \tag{40}$$

The factor $|\cos\alpha|$ has been inserted since the phase α depends sensitively on the spatial pattern. It is nevertheless reasonable to assume, in general, that this factor is of order

† The value $\bar{f}^0 = 7\times10^{-3}$ quoted in the captions to figures (13) and (14) of Ahlers *et al.* (1981) is in error, as can be seen from figure 13 itself. The correct value was used in all calculations.

unity, so that (4) yields an estimate which is close to the experimental value $\bar{f}^1 \approx 0.014$. In view of the uncertainties in both experiment and theory the precise numbers are not significant, but it is gratifying that an *a priori* estimate of this rather subtle effect leads to a numerical value of the correct order of magnitude.

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